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Maximum loads of imperfect systems corresponding to stable bifurcation

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Abstract

A simple and computationally inexpensive approach is presented for obtaining the maximum load factor of an elastic structure considering reduction of load-carrying capacity due to inevitable initial imperfections. The structure has a stable bifurcation point if no initial imperfection exists. An antioptimization problem is formulated for minimizing the maximum loads reduced by the most sensitive imperfection within the convex bounds on the imperfections of nodal locations and nodal loads. The maximum loads may be defined by bifurcation points or deformation constraints. A problem of simultaneous analysis and design with energy method is formulated to avoid laborious nonlinear path-following analysis. The stable bifurcation point is located by minimizing the load factor under constraint on the lowest eigenvalue of the stability matrix. It is shown in the examples that a minor imperfection that is usually dismissed is very important in evaluating the maximum load of a flexible structure. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Buckling analysis; Imperfection sensitivity; Minor imperfection; Antioptimization; Convex model; Simultaneous analysis and design

1. Introduction

The lower bound of the maximum load factor of a geometrically nonlinear structure that exhibits bifurcation-type instability may be evaluated based on the most critical mode of imperfection that maximizes the reduction of the load carrying capacity under constraint on the norm of the imperfection. There have been several studies for finding the most critical mode of imperfections for simple and coincident unstable symmetric bifurcation points (Ho, 1974; Ikeda and Murota, 1990) based on a perturbation approach (Koiter, 1945; Thompson and Hunt, 1973). For a symmetric structure subjected to symmetric proportional loads, which is called *symmetric system* for brevity, an antisymmetric imperfection is classified as major imperfection or first-order imperfection in the sense that the imperfection has direct effect on the derivative of the total potential energy in the direction of the buckling mode. Ohsaki et al. (1998) presented an

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optimization method considering the reduction of maximum load factor due to the most critical mode of major imperfection.

It is laborious, however, to find the buckling load factor by a nonlinear path-following analysis and to obtain the most critical imperfection based on a perturbation approach because the formulations are complicated and are difficult to implement in a finite element analysis program. In addition to this difficulty, the maximum load factor is estimated at the perfect system using sensitivity informations; i.e. a moderately large imperfection is not considered.

Ohsaki (2000) presented an algorithm for obtaining optimum designs of symmetric systems with coincident critical points, and showed that optimization does not always increase imperfection sensitivity. For a symmetric system, a symmetric imperfection is classified as minor imperfection or second-order imperfection (Roorda, 1968) in the sense that the imperfection does not have direct effect on the derivative of the total potential energy in the direction of the buckling mode. For a minor imperfection, the critical point of an imperfect system remains to be a bifurcation point, which is contrary to the fact that the critical point of an imperfect system corresponding to a major imperfection turns out to be a limit point. Although the imperfection sensitivity of a bifurcation load factor corresponding to a minor imperfection is bounded, Ohsaki (2000) showed that a minor imperfection is sometimes more critical than major imperfections if moderately large imperfection is allowed.

One approach for avoiding difficulty in numerical implementation for obtaining the most critical imperfection is to use a stochastic approach. In this case, however, the probability distribution of initial imperfection should be given appropriately, preferably based on experiments. The convex model is very effective for the case where stochastic approach cannot be used (Ben-Haim and Elishakoff, 1990). It considers uncertainty within known bounds on the parameters. Elishakoff et al. (1994c) applied the convex model to stability analysis of imperfection sensitive columns on elastic foundation, where prebuckling deformation can be neglected in buckling analysis and the buckling load factor is linearized with respect to the imperfection parameters. They compared the results by a stochastic approach and the convex model, and showed that only few modes are necessary for buckling analysis and for modeling imperfections. Elishakoff et al. (1994a) presented a method for obtaining the most critical imperfection for elastic static problem using the antioptimization approach to obtain the possible worst case values of the parameters. They considered uncertainty in loads as well as the nodal locations. Uncertainty in the elastic modulus has been considered in Elishakoff et al. (1994b). Pantelides (1996a) introduced imperfections in geometry and material properties. Pantelides (1996b) used elliptic bounds for buckling analysis of columns on uncertain elastic foundation. A convex model for buckling of bars connected by springs are discussed in Pantelides (1995).

The method called *simultaneous analysis and design*, which is abbreviated as SAND, is very effective for reducing the computational cost for path-following analysis that should be carried out at each step of optimization or antioptimization of geometrically nonlinear structures. It considers the state variables as well as the design variables as independent variables. Haftka (1985) incorporated the equilibrium equations into the objective function by using the interior penalty functions, and presented an efficient approach for avoiding illconditioning of the Hessian of the Lagrangian or the objective function. His approach has been shown to be applicable to truss topology optimization problems (Sankaranarayanan et al., 1994). The method with direct incorporation of the equilibrium equations as equality constraints has also been presented (Wu and Arora, 1987; Orozco and Ghattas, 1997).

Contrary to imperfection-sensitive structures such as cylindrical shells and stiffened plates, the bifurcation point of a column that has a stable postbuckling path disappears due to a small major imperfection; e.g. antisymmetric imperfection of a symmetric system. A question then arises how the maximum load factor of such stable structures should be defined. One approach is to allow deformation along the bifurcation path that has the load factor above the bifurcation load (Pietrzak, 1996). In this case, the maximum load factor may be determined by the constraints on displacements and/or stresses. The antisymmetric component of initial imperfection, however, may happen to be very small, and sudden deformation may occur near the bifurcation point. Therefore in some situations it may be unsafe to allow loading along the bifurcation path, and the bifurcation load factor should be used for defining the maximum load.

In this paper, a simple and numerically inexpensive approach is presented for determining the maximum load factors of imperfect elastic structures considering imperfections of nodal locations and nodal loads. It is shown in the examples that the reduction of the maximum loads defined by displacement constraints is very small if a major imperfection is considered and moderately large displacements are allowed. Therefore, minor imperfections should be considered in defining the most critical mode of imperfection. An antioptimization problem is formulated so as to minimize the bifurcation load factor within the convex bounds on the imperfection parameters. A relaxed problem is solved based on the SAND, where the bifurcation load is determined by minimizing the load factor under constraint on the lowest eigenvalue of the stability matrix allowing imperfections of nodal loads. This way, laborious nonlinear path-following analysis is successfully avoided. It is shown in the examples of a 20-bar truss that the most critical mode of minor imperfection can be successfully obtained by the proposed approach.

2. Maximum load factor of an imperfect system

Consider a finite dimensional elastic structure subjected to quasi-static proportional loads $\boldsymbol{P}$ defined by the load factor Λ as $\boldsymbol{P} = \Lambda \boldsymbol{p}$, where $\boldsymbol{p}$ is the specified vector of load pattern. The vector of nodal displacements is denoted by $\boldsymbol{U} = \{U_i\}$. Let ξ denote an imperfection parameter that represents any type of imperfection including initial dislocation of nodes, distortion of cross-sectional shape of a member, etc. The total potential energy $\Pi(\boldsymbol{U}, \Lambda; \xi)$ is defined as

$$\Pi(\boldsymbol{U}, \Lambda; \xi) = H(\boldsymbol{U}; \xi) - \Lambda \boldsymbol{p}^T(\xi)\boldsymbol{U} \tag{1}$$

where $H(\boldsymbol{U}; \xi)$ is the strain energy which is assumed not to depend explicitly on Λ. This assumption is valid for a proportionally loaded structure modeled by a finite element formulation using nodal displacements as variables for defining deformation.

The equivalent nodal force $\boldsymbol{F}(\boldsymbol{U}; \xi) = \{F_j(\boldsymbol{U}; \xi)\}$ is defined by

$$F_j(\boldsymbol{U}; \xi) = \frac{\partial H}{\partial U_j} \quad (j = 1, 2, \ldots, n) \tag{2}$$

where n is the number of degrees of freedom. The equilibrium equations are written as

$$F_j(\boldsymbol{U}; \xi) = \Lambda p_j(\xi) \quad (j = 1, 2, \ldots, n) \tag{3}$$

In the following, the arguments $\boldsymbol{U}$, Λ and ξ are omitted except the case where dependence on those variables is important. The stability matrix $\boldsymbol{S}$, which is the tangent stiffness matrix used for nonlinear analysis, is given as

$$\boldsymbol{S} = \left[\frac{\partial^2 H}{\partial U_i \partial U_j} \right] \tag{4}$$

The rth eigenvalue λ_r and eigenvector $\boldsymbol{\Phi}_r$ of $\boldsymbol{S}$ are obtained from

$$\boldsymbol{S}\boldsymbol{\Phi}_r = \lambda_r \boldsymbol{\Phi}_r, \quad (r = 1, 2, \ldots, n) \tag{5}$$

The critical load factor Λ^{c} corresponds to $\lambda_1 = 0$, where λ_1 is the lowest eigenvalue.

Define α as

$$\alpha = \sum_{i=1}^{n} \frac{\partial^2 \Pi}{\partial \xi \partial U_i} \Phi_{1i} \tag{6}$$

where Φ_{1i} is the ith component of $\boldsymbol{\Phi}_1$. The major and minor imperfections are characterized by $\alpha \neq 0$ and $\alpha = 0$, respectively (Roorda, 1968). For a symmetric system, the prebuckling deformation is symmetric and the bifurcation mode is antisymmetric. In this case, a symmetric and antisymmetric imperfections correspond to minor and major imperfections, respectively. Figs. 1 and 2 illustrate the relation between Λ and a representative antisymmetric displacement component U for the cases of major and minor imperfections, respectively. It is seen from Fig. 1 that Λ increases above the bifurcation load factor Λ^{c} of the perfect system if a major imperfection exists. For the case of minor imperfection, as shown in Fig. 2, an imperfect system still has a bifurcation point, and the bifurcation load factor may increase or decrease depending on the sign of the imperfection parameter.

Since antisymmetric components of deformation along the bifurcation path of the stable bifurcation point may be very large, the maximum load should be defined in view of the stresses and/or displacements. A question arises whether it is safe to expect loads above the bifurcation load. Although a critical point does not exist for a structure with a major imperfection, the imperfection may happen to be extremely small, and the structure may reach a bifurcation point that causes sudden dynamic antisymmetric mode of deformation. Therefore, the maximum load factor of a structure exhibiting stable bifurcation may be defined by either of the following criteria:

C1 Bifurcation load factor.

C2 Load factor corresponding to the specified limits on stresses and/or displacements.

Consider a case where the maximum load factor is defined by the upper bound $\bar{U}$ of the displacement component U for a system illustrated in Figs. 1 and 2. It is observed from Fig. 1 that the magnitude of reduction of the maximum load factor due to a major imperfection is very large for a small range of $\bar{U}$, e.g., $\bar{U} = U_a$, but it decreases as $\bar{U}$ is increased to, e.g., $\bar{U} = U_b$. For a minor imperfection, the magnitude of reduction does not strongly depend on $\bar{U}$, and it is larger than that to a major imperfection if $\bar{U}$ is moderately large, e.g., $\bar{U} = U_b$. Therefore, for the case where C2 is used, the most critical mode of imperfection will be a major imperfection if $\bar{U}$ is sufficiently small, otherwise the maximum load factor should be defined by a minor imperfection. If C1 is used, the most critical imperfection should be a minor imperfection because the bifurcation point disappears if a major imperfection exists. In this paper, we consider a flexible

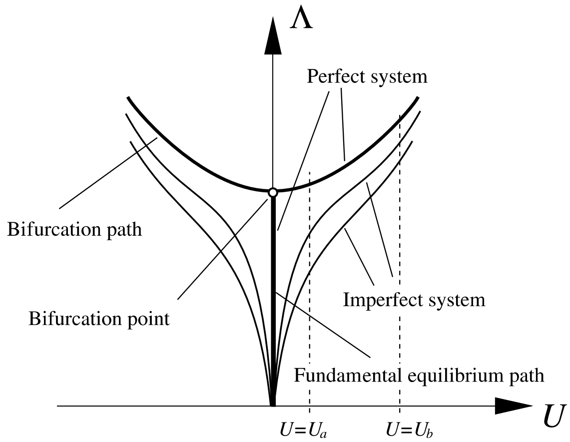

Fig. 1. Equilibrium paths of perfect and imperfect systems corresponding to major imperfection.

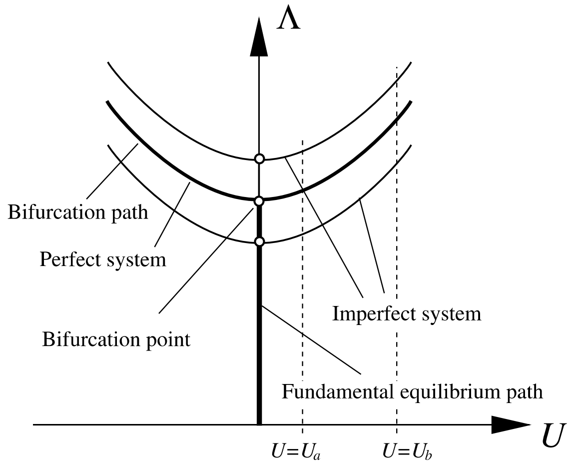

Fig. 2. Equilibrium paths of perfect and imperfect systems corresponding to minor imperfection.

system allowing a moderately large deformation. In this case, minor imperfection plays a key role in evaluating the maximum load factors of the imperfect systems considering both criteria C1 and C2. A good approximate maximum load factor and the corresponding most critical mode of imperfection will be found by minimizing the bifurcation load factor even for the case of C2.

Summarizing the discussion above, we define the most critical imperfection by reduction of the bifurcation load factor due to minor imperfections based on the following reasons:

(1) Even for a stable bifurcation, reaching the bifurcation point should be avoided because it leads to a sudden dynamic deformation. Since the bifurcation point disappears if a major imperfection exists, the most critical imperfection for this case is a minor imperfection.

(2) Since we consider a flexible structure and allow moderately large deformation, the maximum load defined by deformation constraints is dramatically reduced by minor imperfections rather than major imperfections, and sensitivity of the maximum load is almost equivalent to that of the bifurcation point.

3. Antioptimization problem

If we consider only symmetric systems, imperfections can easily be divided into major and minor imperfections based on the symmetry conditions. For a more complicated structure without explicit symmetry properties subjected to nonsymmetric loads, classification of imperfection is not straightforward. In this section, a method is presented for obtaining most critical minor imperfection without carrying out any preprocessing for orthogonalization or classification of imperfection modes. Note that the most critical major imperfection of an unstable symmetric bifurcation point may be found directly by the perturbation approaches (Ho, 1974; Ikeda and Murota, 1990). There has been no study, however, for finding most critical imperfection of a stable symmetric bifurcation point.

Let $\boldsymbol{\xi}_i$ $(i = 1, 2, \dots, m)$ denote the vector of ith set of imperfection parameters including any possible type of imperfection such as nodal locations and cross-sectional areas. The norm of $\boldsymbol{\xi}_i$ is denoted by $e_i(\boldsymbol{\xi}_i)$ which is a convex function of $\boldsymbol{\xi}_i$. Suppose an upper bound $\bar{e}_i$ is given for $e_i(\boldsymbol{\xi}_i)$ by an approach similar to that of the convex model (Ben-Haim and Elishakoff, 1990). In the formal convex model, the objective function is linearized by utilizing the first order sensitivity information, and the optimal or antioptimal solution is uniquely determined. In this paper, however, the nonlinear buckling load factor is directly used as objective function in order to rigorously incorporate the geometrical nonlinearity.

The set of vectors ξ_i is divided into major imperfections ξ_i^{I} and minor imperfections ξ_i^{II}. The values corresponding to major and minor imperfections are indicated by superscripts $(\ )^{\mathrm{I}}$ and $(\ )^{\mathrm{II}}$, respectively; e.g. the upper bound for $e_i^{\mathrm{II}}(\xi_i^{\mathrm{II}})$ is denoted by $\bar{e}_i^{\mathrm{II}}$. Let ξ^{II} denote the vector that consists of all the elements of the vectors ξ_i^{II}, $(i = 1, 2, \ldots, m^{\mathrm{II}})$. The maximum load of the imperfect system considering reduction due to the most critical mode of minor imperfection is defined as the solution of the following optimization problem:

$$\mathrm{P1}: \quad \text{minimize} \quad \Lambda^{\mathrm{c}}(\xi^{\mathrm{II}}) \tag{7}$$

$$\text{subject to} \quad e_i^{\mathrm{II}}(\xi_i^{\mathrm{II}}) \leqslant \bar{e}_i^{\mathrm{II}} \quad (i = 1, 2, \ldots, m^{\mathrm{II}}) \tag{8}$$

This type of problem for finding the minimum load factor is called antioptimization problem (Elishakoff et al., 1994a). The variables in P1 are ξ_i^{II}, $(i = 1, 2, \ldots, m^{\mathrm{II}})$.

As noted above, the objective function of P1 is not linearized with respect to the imperfection parameters; i.e. the rigorous nonlinear formulation is used for Λ^{c}. P1 may be solved by using an appropriate gradient-based optimization algorithm if sensitivity coefficients of the critical load factors can be found (Ohsaki, 2000). However, $\Lambda^{\mathrm{c}}(\xi^{\mathrm{II}})$ corresponding to the given set of ξ_i^{II} should be determined by tracing the fundamental equilibrium path at each iterative step of optimization. Therefore, the formulation of P1 is computationally expensive.

If we fix ξ^{II} and only consider major imperfections, the region in the $(\Lambda - U)$-space where $\lambda_1 \leqslant 0$ is satisfied is as indicated by *feasible region* in Fig. 3, where U is a representative generalized displacement generated due to existence of a major imperfection. For a symmetric system, U represents an antisymmetric component of deformation. The thick curve in Fig. 3 is the bifurcation path of the perfect system, and thin curves are equilibrium paths of imperfect systems. The dotted curves indicate unstable equilibrium points. Note that the region bounded by the dashed curve ABC is feasible for the constraint $\lambda_1 \leqslant 0$. Since we consider the case where the perfect system exhibits stable bifurcation, the feasible region in the vicinity of the bifurcation point is convex with respect to U and Λ. Hence, the buckling load factor is found by minimizing Λ with respect to ξ_i^{I} under constraint of $\lambda_1 \leqslant 0$. We further minimize Λ with respect to ξ_i^{II} to obtain the most sensitive imperfection. Since both processes correspond to minimization of Λ, those can be carried out simultaneously.

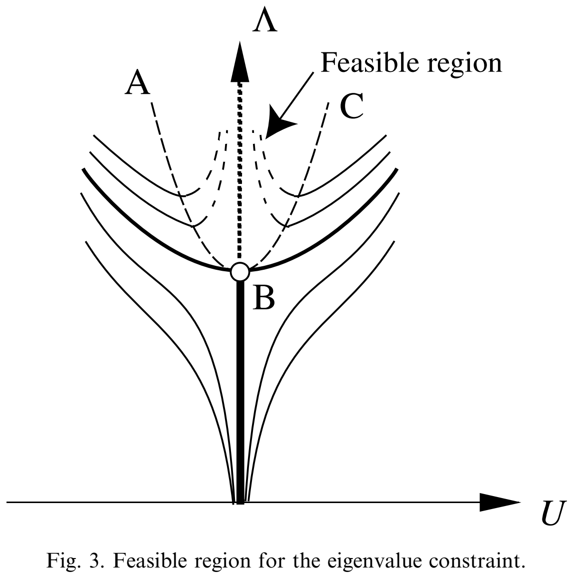

Fig. 3. Feasible region for the eigenvalue constraint.

In order to reduce the computational cost for geometrically nonlinear path-following analysis, nodal displacements are also considered as variables of the optimization problem, and analysis and optimization are simultaneously carried out. If we use the most simple formulations of SAND, the state variables $\boldsymbol{U}$ are iteratively updated to satisfy the equality constraints. The optimization problem with equality constraints, however, are computationally costly. A two stage algorithm can be used for reducing the computational cost (Wu and Arora, 1987). In this case, a special algorithm should be implemented for optimization. If the equality constraints are included in the objective function as penalty terms (Haftka, 1985), the optimization problem can be solved by simply using an optimization package.

In this study, we consider $\boldsymbol{U}$ as variables which are the same level as ξ^{II} and relax P1 allowing imperfections in nodal loads. Therefore, we do not need the exact equilibrium state corresponding to the *perfect* nodal loads. The ranges of the nodal loads are given as

$$\Lambda p_j^{\mathrm{L}} \leqslant P_j \leqslant \Lambda p_j^{\mathrm{U}} \quad (j = 1, 2, \ldots, n) \tag{9}$$

where p_j^{L} and p_j^{U} are the specified lower and upper bounds, respectively. Suppose the case where the upper bound ΔP for the error in the load is proportional to Λ as $\Delta P = \Lambda \Delta p$. Eq. (9) is then written as

$$\Lambda(p_j - \Delta p) \leqslant P_j \leqslant \Lambda(p_j + \Delta p) \quad (j = 1, 2, \ldots, n) \tag{10}$$

Let $\boldsymbol{\xi}$ denote the vector consisting of $\boldsymbol{\xi}_i$ including minor and major imperfections. From Eq. (2), the internal nodal forces $\boldsymbol{F}^*(\boldsymbol{U}; \boldsymbol{\xi}) = \{F_j^*(\boldsymbol{U}; \boldsymbol{\xi})\}$ equivalent to the displacements $\boldsymbol{U}$ of an imperfect system are defined by

$$F_j^*(\boldsymbol{U}; \boldsymbol{\xi}) = \frac{\partial H(\boldsymbol{U}; \boldsymbol{\xi})}{\partial U_j} \quad (j = 1, 2, \ldots, n) \tag{11}$$

where $(\ )^*$ indicates a function of $\boldsymbol{U}$ and $\boldsymbol{\xi}$. F_j^* is then calculated for each trial displacement vector during the optimization process.

The optimization problem to be solved is formulated as follows for finding the minimum value of Λ under constraints on the norms of imperfections and the lowest eigenvalue $\lambda_1^*(\boldsymbol{U}, \boldsymbol{\xi})$ of the stability matrix:

$$\mathrm{P2}: \quad \text{minimize} \quad \Lambda \tag{12}$$

$$\text{subject to} \quad e_i(\boldsymbol{\xi}_i) \leqslant \bar{e}_i \quad (i = 1, 2, \ldots, m) \tag{13}$$

$$\Lambda(p_j - \Delta p) \leqslant F_j^*(\boldsymbol{U}; \boldsymbol{\xi}) \leqslant \Lambda(p_j + \Delta p) \quad (j = 1, 2, \ldots, n) \tag{14}$$

$$\lambda_1^*(\boldsymbol{U}; \boldsymbol{\xi}) \leqslant 0 \tag{15}$$

The variables of this problem are $\boldsymbol{U}$, $\boldsymbol{\xi}$ and Λ. Only computation of $F_i^*(\boldsymbol{U}; \boldsymbol{\xi})$ and $\lambda_1^*(\boldsymbol{U}; \boldsymbol{\xi})$ is needed for the current value of $\boldsymbol{U}$ and $\boldsymbol{\xi}$ at each iterative step of optimization, and the laborious path-following analysis is not needed.

4. Examples

Consider a column-type 20-bar plane truss as shown in Fig. 4. The lengths of members in x- and y-directions are 100 and 200 cm, respectively. The cross-sectional areas are 2.0 cm^2 for all the truss members. The proportional loads in the negative y-direction at nodes 7 and 8 are given as Λp, where $p = 98$ kN. The elastic modulus is 205.8 GPa. The axial strain is defined by the Green's strain. Optimization is carried out by IDESIGN Ver. 3.5 (Arora and Tseng, 1987), where the sequential quadratic programming is used, and the gradients of the objective and constraint functions are computed by the finite difference approach. Very strict convergence criteria have been assigned for obtaining rigorous optimal solutions; i.e. the constraint violation is limited to 1.0×10^{-5}, and the difference of the objective values in two consecutive steps should

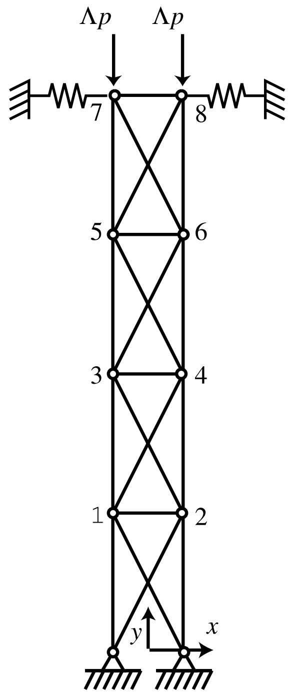

Fig. 4. A column-type 20-bar plane truss.

be less than 1.0×10^{-6}. The computational cost will be reduced if larger limits are used in practical situation. Computation has been carried out on a personal computer with AMD Athron 1.0 GHz.

The vector ξ of the imperfection parameters consists of the coordinates of all the nodes except two supports. Note that ξ includes minor and major imperfections. Therefore the size of ξ is equal to n. The norm $\tilde{e}(\xi)$ of the imperfection is defined as

$$\tilde{e}(\xi) = \frac{1}{n}\sqrt{\xi^T \xi} \tag{16}$$

The upper bound Δp for the error in the nodal loads is 1% of p. Note that the imperfection of loads is assumed to exist in all the displacement components. The number of variables ξ, $\boldsymbol{U}$ and Λ in P2 is 33. The extensional stiffness of each spring attached at nodes 7 and 8 is denoted by κ. We consider two cases with $\kappa = 0$ and $\kappa = 102.9$ kN/m which are referred to as *column-type truss* and *laterally supported truss*, respectively.

Let $\boldsymbol{\Phi}^{\mathrm{A}}$ and $\boldsymbol{\Phi}^{\mathrm{S}}$ denote the lowest antisymmetric and symmetric linear buckling modes, respectively, of the perfect system. Imperfection sensitivity properties are first investigated for imperfections in the directions of $\boldsymbol{\Phi}^{\mathrm{A}}$ and $\boldsymbol{\Phi}^{\mathrm{S}}$ which correspond to major and minor imperfections, respectively. Imperfection modes may also be defined by the eigenmodes $\boldsymbol{\Phi}_r$ of $\boldsymbol{S}$ at the critical point. Since the bifurcation mode $\boldsymbol{\Phi}_1$ is antisymmetric and the lowest symmetric mode of $\boldsymbol{S}$ does not have any physical meaning, and since the prebuckling deformation is not very large for a perfect column-type trusses, it is reasonable to define the imperfection by the linear buckling modes.

The upper bounds 100 and 300 cm are given for the absolute values of the components of ξ and $\boldsymbol{U}$, respectively. Moderately large upper bounds should be given for the components of ξ and $\boldsymbol{U}$ so as to exclude deformation above possible local maxima of the bifurcation path in the process of optimization, even if those constraints are inactive at the optimal solution.

4.1. A column-type truss

Consider the column-type truss without springs; i.e. $\kappa = 0$. The critical load factor of the perfect system is 3.9366, where the buckling mode is antisymmetric with respect to the y-axis and the critical point is a symmetric bifurcation point.

We first investigate imperfection sensitivity of the maximum load factor in the directions of $\boldsymbol{\Phi}^{\mathrm{A}}$ and $\boldsymbol{\Phi}^{\mathrm{S}}$, respectively, which are as shown in Fig. 5(a) and (b). Note that the imperfection mode $\Delta\boldsymbol{p}$ of the nodal loads is also considered in the same directions as the nodal imperfections, where $\Delta\boldsymbol{p}$ is scaled so that its maximum absolute value is equal to 1% of p. Fig. 6 shows the relation between the horizontal displacement δ of node 8 and the load factor for three cases of perfect and imperfect systems in the direction of $\boldsymbol{\Phi}^{\mathrm{A}}$ with $\tilde{e}(\xi) = 1.0$ and 5.0 cm. It is observed from Fig. 6 that Λ slightly increases along the bifurcation path, and the critical point of the perfect system is a stable symmetric bifurcation point. Fig. 7 shows the relation between δ and Λ for three cases of perfect and imperfect systems for minor imperfection corresponding to $\boldsymbol{\Phi}^{\mathrm{S}}$ with $\tilde{e}(\xi) = 1.0$ and 5.0 cm.

Suppose the maximum load factor Λ^{M} is defined by the displacement constraint $\delta \leqslant \bar{\delta}$. It may be observed from Figs. 6 and 7 that the reduction of Λ^{M} due to a major imperfection is larger than that to a minor imperfection if $\bar{\delta}$ is small, but a minor imperfection dominates if $\bar{\delta}$ is sufficiently large. For instance, if $\tilde{e}(\xi) = 5.0$ cm, reduction in the direction of $\boldsymbol{\Phi}^{\mathrm{S}}$ is larger than that of $\boldsymbol{\Phi}^{\mathrm{A}}$ in the range $\delta > 179$ cm. The important property observed in Fig. 7 is that the magnitude of reduction of Λ^{M} does not strongly depend

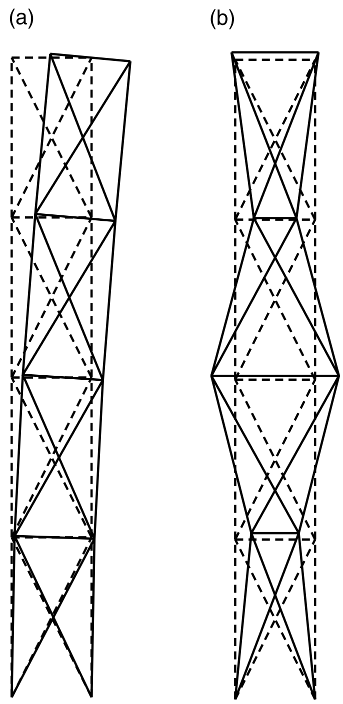

Fig. 5. Lowest symmetric and antisymmetric linear buckling modes of the column-type truss: (a) antisymmetric mode and (b) symmetric mode.

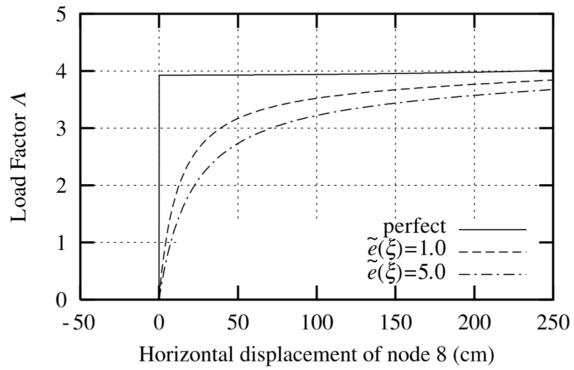

Fig. 6. Relation between δ and Λ for perfect and imperfect systems of the column-type truss in the direction of $\boldsymbol{\Phi}^{\mathrm{A}}$.

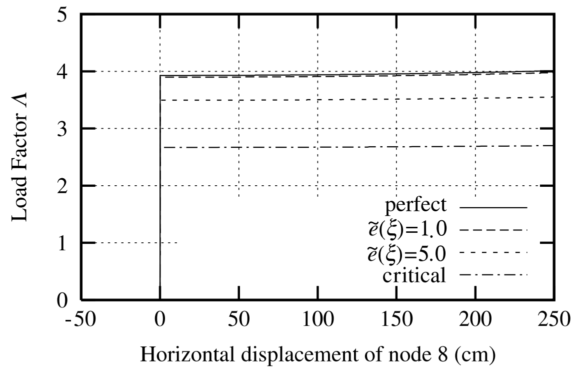

Fig. 7. Relation between δ and Λ for perfect and imperfect systems of the column-type truss in the direction of $\boldsymbol{\Phi}^{\mathrm{S}}$.

on the value of $\bar{\delta}$. Therefore, the most critical mode of minor imperfection may be successfully obtained by solving P2 considering only the bifurcation load factor.

The minimum value of Λ of P2 for $\tilde{e}(\xi) \leqslant \bar{e} = 5.0$ cm is 2.6693 which is about 68% of $\Lambda^{\mathrm{c}} = 3.9366$ of the perfect system. The most critical mode of nodal imperfection ξ^{M} is symmetric as shown in Fig. 8(a), where the initial value for ξ has been given as $\boldsymbol{\Phi}^{\mathrm{S}}$. The displacements and load factor at buckling of the perfect system have been assigned to the initial values of $\boldsymbol{U}$ and Λ, respectively. Most critical imperfections of nodal locations and nodal loads are also listed in Table 1. It is observed from Table 1 that all the components of $\Delta\boldsymbol{p}$ are equal to the upper or lower bound.

The number of optimization steps, CPU time and the optimal objective value are as listed in the first row of Table 2. Note that Λ^{c} of the imperfect system corresponding to $\tilde{e}(\xi) = 5.0$ cm in the direction of $\boldsymbol{\Phi}^{\mathrm{S}}$ is 3.4747 which is larger than that for ξ^{M}. Therefore, $\boldsymbol{\Phi}^{\mathrm{S}}$ cannot be used as an approximation for ξ^{M}. Although the number of steps is considerably large, an almost optimal solution has been found within 30 steps. The relation between δ and Λ for the most critical case is also plotted in Fig. 7.

Since the structure and loading conditions considered here have obvious symmetry properties, it is very easy to divide ξ and $\boldsymbol{U}$ into symmetric and antisymmetric components. If we consider only symmetric components of ξ and $\boldsymbol{U}$, the maximum load factor of the symmetric system is 2.6693 which agrees within the accuracy of five digits with the value obtained by the formulation including asymmetric imperfections and deformations. The mean absolute value of deviation of ξ^{M} from those of the symmetric system is

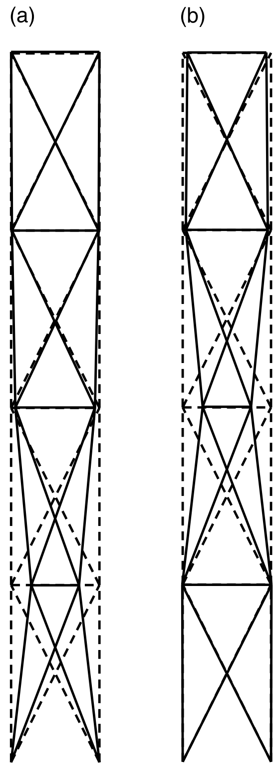

Fig. 8. Most critical modes of imperfection: (a) column-type truss and (b) laterally supported truss.

Table 1
Most critical imperfections of nodal locations and nodal loads

Node	Direction	Location (cm)	Load $(\Delta p_i/p)$
1	x	13.781	0.01
	y	−0.22715	−0.01
2	x	−13.783	−0.01
	y	−0.22766	−0.01
3	x	2.9410	0.01
	y	0.22275	−0.01
4	x	−2.9375	−0.01
	y	0.22211	−0.01
5	x	0.68235	0.01
	y	0.013910	−0.01
6	x	−0.68769	−0.01
	y	0.011772	−0.01
7	x	−0.049643	0.01
	y	0.91936	−0.01
8	x	0.049153	−0.01
	y	0.91907	−0.01

4.3335×10^{-3} which is very small compared to the maximum absolute value 13.7810 of ξ^{M}. The computational cost is reduced as shown in the second row of Table 2 if we consider only symmetric imperfections and deformations.

Table 2
Number of iteration steps, CPU time and the objective value for the column-type truss

	Number of steps	CPU time (s)	Objective value
Symmetric initial solution	38	36.8	2.6693
Symmetric imperfection	14	6.7	2.6693
Asymmetric initial solution	27	25.2	2.6693
Linear strain	29	26.5	2.6864
Incremental analysis	67	54.2	2.6708

If we do not exclude major imperfections and the initial values of all the components of nodal imperfections and nodal displacements are equal to 0.1 which are not symmetric, the deviation from the symmetric solution increases to 3.4390×10^{-2}, but the deviation is still very small. The number of steps and CPU time for this case are listed in the third row of Table 2. Note that the computational cost is smaller than that from the symmetric initial solution. Therefore, symmetricity of initial solution does not always lead to reduction of computational cost, but usually leads to a rigorously symmetric solution. It should be noted that the optimal objective values are same up to five digits for several cases we tested from different initial solutions.

The maximum load factor obtained by linearizing the equilibrium equation (11) with respect to $\boldsymbol{U}$ is 2.6864. Therefore, the prebuckling deformation may be neglected for the column-type truss as this example. The convergence property, however, does not improve as the result of neglecting the geometrical nonlinearity as observed from the fourth row of Table 2. In general cases including dome-type structures, prebuckling deformation cannot be neglected, and the geometrically nonlinear formulation presented in this paper should be used.

Problem P1 has been directly solved for comparison purpose, where path-following analysis is to be carried out to evaluate the bifurcation load factor at each step of optimization. Only symmetric imperfections are considered. If asymmetric imperfection exists, the bifurcation point disappears and the optimization process obviously does not converge. Finite difference method has been used for sensitivity analysis, and the equilibrium path is traced by the displacement increment method. The computational results are listed in the last row of Table 2 which should be compared to the second row because only symmetric imperfections are considered here. It is observed from Table 2 that computational cost for P1 is very large compared to that for P2. Computational cost, however, strongly depends on the methods of path-following analysis and optimization. The ratio of CPU time for P1 to that of P2 will be different if analytical sensitivity analysis is used instead of finite difference approach. However, the cost for evaluating the constraint functions for P2 is very small because the equivalent nodal loads are obtained by an algebraic computation and λ_1 is computed by carrying out eigenvalue analysis only once which is not costly compared to the path-following analysis. Since imperfections on $\boldsymbol{p}$ are also considered in P1, the number of variables for P1 and P2 for this case are 32 and 33, respectively, which are almost same. Therefore, it can be concluded that the computational cost for P2 is generally smaller than that for P1.

4.2. A laterally supported truss

Consider next a laterally supported truss with $\kappa = 102.9$ kN/m. The ratio of the extensional stiffness of the spring to that of the horizontal truss member is 0.005. The buckling load factor of the perfect system is 15.497, where the buckling mode is antisymmetric with respect to y-axis. Therefore the critical point is a symmetric bifurcation point.

Fig. 9 shows the relation between δ and Λ for three cases of perfect and imperfect systems corresponding to a major imperfection $\boldsymbol{\Phi}^{\mathrm{A}}$ with $\tilde{e}(\boldsymbol{\xi}) = 1.0$ and 5.0 cm. It is seen from Fig. 9 that the critical point of the perfect system is a stable bifurcation point, and the critical point of imperfect systems are limit points that

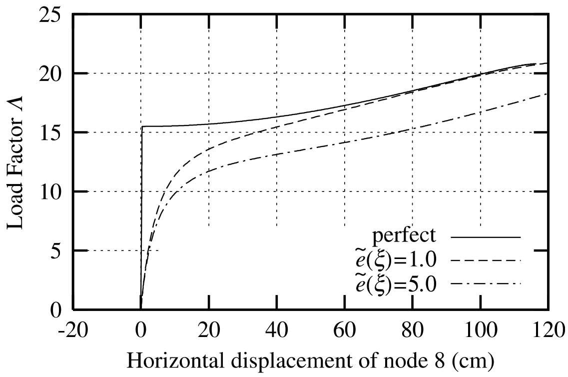

Fig. 9. Relation between δ and Λ for perfect and imperfect systems of the laterally supported truss in the direction of $\boldsymbol{\Phi}^{\mathrm{A}}$.

are far above the bifurcation point. Note that the reduction of Λ^{M} is very small for $\tilde{e}(\boldsymbol{\xi}) = 1.0$ cm in the moderately large range of displacement. Variation of λ_1 with respect to Λ for $\tilde{e}(\boldsymbol{\xi}) = 1.0$ cm is as shown in Fig. 10. The lowest eigenvalue has a local minimum near the bifurcation point, but increases to positive values before reaching 0 at the limit point.

Fig. 11 shows the relation between δ and Λ for three cases of perfect and imperfect systems for minor imperfection in the direction of $\boldsymbol{\Phi}^{\mathrm{S}}$ with $\tilde{e}(\boldsymbol{\xi}) = 1.0$ and 5.0 cm. The equilibrium paths are plotted up to the second and first critical point, respectively, for perfect and imperfect systems. Note that the bifurcation load factor for $\tilde{e}(\boldsymbol{\xi}) = 5.0$ cm is 14.107, where imperfection in nodal loads are also considered in the direction of $\boldsymbol{\Phi}^{\mathrm{S}}$.

The optimal value of Λ obtained by solving P2 for $\tilde{e}(\boldsymbol{\xi}) \leqslant \bar{e} = 5.0$ cm is 12.483 which is about 81% of $\Lambda^{\mathrm{c}} = 15.497$ of the perfect system, where the initial value for $\boldsymbol{\xi}$ has been given as $\boldsymbol{\Phi}^{\mathrm{S}}$. The displacements and load factor at buckling of the perfect system have been assigned to the initial values of $\boldsymbol{U}$ and Λ, respectively. The relation between δ and Λ for $\boldsymbol{\xi}^{\mathrm{M}}$ which is symmetric as shown in Fig. 8(b) is also plotted in Fig. 9. In this case the reduction of Λ^{c} due to the most critical imperfection is much larger than that to the imperfection with the same norm in the direction of $\boldsymbol{\Phi}^{\mathrm{S}}$. Note from Fig. 8(a) and (b) that $\boldsymbol{\xi}^{\mathrm{M}}$ strongly depends on the extensional stiffness of the spring.

If we consider only symmetric components for $\boldsymbol{\xi}$ and $\boldsymbol{U}$, the maximum load factor of symmetric system is 12.483 which agrees within the accuracy of five digits with the value obtained by the formulation including

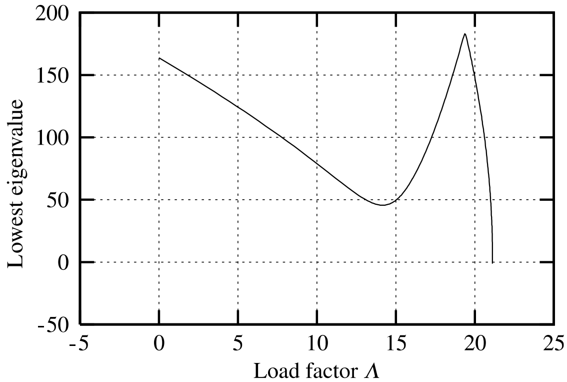

Fig. 10. Relation between load factor and the lowest eigenvalue with $\tilde{e}(\boldsymbol{\xi}) = 1.0$ cm.

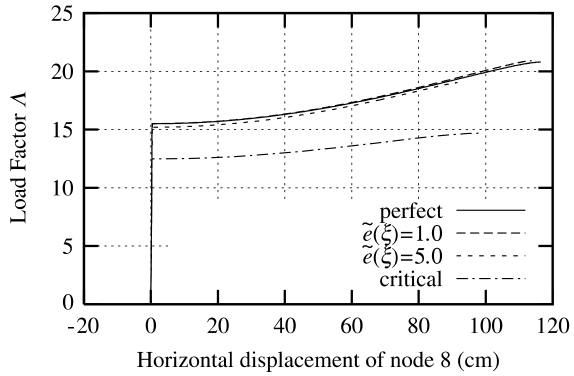

Fig. 11. Relation between δ and Λ for perfect and imperfect systems of the laterally supported truss in the direction of $\boldsymbol{\Phi}^{\mathrm{S}}$.

asymmetric imperfections and deformations. The mean absolute value of deviation of ξ^{M} from the most critical imperfection mode of the symmetric system is 2.0182×10^{-2} which is very small compared to the maximum absolute value 13.7810 of ξ^{M}. If we start from a nonsymmetric initial values such that all the components of $\boldsymbol{\xi}$ and $\boldsymbol{U}$ are equal to 0.1, the deviation increases to 6.5855×10^{-2} which is still a small value. The maximum load factor for linear case is 12.863. Also for this case, convergence property did not improve as a result of neglecting the effect of prebuckling deformation.

5. Conclusions

A simple and computationally inexpensive approach has been presented for obtaining the maximum load factor of an elastic structure that has a stable bifurcation point if no initial imperfection exists. In the proposed method, an antioptimization problem is first formulated for minimizing the load factor within the convex region of possible imperfections. The problem is then relaxed and reformulated by using the SAND as well as the energy based approach to obtain the most critical minor imperfection and the corresponding bifurcation load factor also considering the imperfection in nodal loads. The bifurcation load factor is located as a minimum value of the load factor with respect to the major imperfections under constraint on the lowest eigenvalue of the stability matrix. The variables of the problem are the imperfection parameters, nodal displacements and the load factor, and laborious nonlinear analysis for tracing equilibrium path is avoided.

The equilibrium paths of perfect and imperfect systems have been investigated for a 20-bar column-type truss with and without lateral springs considering major and minor imperfections of various magnitudes. It has been shown for a flexible structure allowing moderately large displacements that the antisymmetric linear buckling mode cannot always be the most critical mode of imperfection and that a minor imperfection is very important for estimating the reduction of the maximum load factor defined by the displacement constraints. It has also been shown that the reduction of the maximum load factor defined by the displacement constraints does not strongly depend on the value of the upper bound of displacement if a minor imperfection is considered. The possibility of reaching the bifurcation point that leads to sudden dynamic antisymmetric deformation should also be avoided in practical situation. Therefore it is reasonable to define the most critical mode of imperfection in the direction of symmetric minor imperfection for a flexible structure allowing moderately large deformation.

The most critical minor imperfection has been shown to be successfully obtained by solving the proposed antioptimization problem using an appropriate nonlinear programming algorithm. The antioptimal solutions have been found under several problem settings, and it has been confirmed that the proposed method has advantages over the method with path-following analysis in view of computational cost and convergence property.

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